Maxwell's Equations

Some vector relationships. S scalar, V vector.

∇ del operator

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

grad S = ∇ S = $i\frac{\partial S}{\partial x} + j\frac{\partial S}{\partial y} + k\frac{\partial S}{\partial z}$

div V = $\nabla \cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

curl V = ∇ x V = $\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)i + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)j + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)k$

div grad S = $\nabla \cdot \nabla S = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\left(i\frac{\partial S}{\partial x} + j\frac{\partial S}{\partial y} + k\frac{\partial S}{\partial z}\right)$

$$= \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} = \nabla^2 S$$

$\nabla^2 S$ is the Laplacian operator on a scalar

Laplacian operator on a vector is

$\nabla^2 V = i\,\nabla^2 V_x + j\,\nabla^2 V_y + k\,\nabla^2 V_z$

The curl curl V operation . ∇ x ∇ x V

Consider just the x component

$\text{curl}_x \text{ curl } V = \left[\frac{\partial}{\partial y}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\right] i$

$= \left[\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x\,\partial y} + \frac{\partial^2 V_z}{\partial x\,\partial z}\right] i - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x\,\partial y} + \frac{\partial^2}{\partial x\,\partial z}\right] V_x\, i$

$= \nabla_x(\nabla\cdot V) - \nabla^2 V_x$

Similarly for the y and z components hence

curl curl V = ∇ x ∇ x V = $\nabla(\nabla\cdot V) - \nabla^2 V$

Two Useful Theorems

Gauss's Theorem

$\iint V\, ds = \iiint \text{div } V\, dv$

Stokes's Theorem

$\oint V \cdot dl = \iint \text{curl } V\, ds = \iint \nabla \times V\, ds$

Electrical Laws

Gauss’ Law

$\nabla \cdot D = \rho_v$ 1

$\int\int\int(\nabla \cdot D)\, dv = \int\int\int \rho_v\, dv$

$\iint D\, ds = q$ by Gauss’s Theoreom

Gauss’ Magnetism Law (no free poles)

$\nabla \cdot B = 0$ 2

Faraday’s Law

$\phi(t) = \int\int B(t)\, ds$

$e = -\frac{d\phi}{dt}$

$= -\frac{d}{dt}\int\int B(t)\, .\, ds = -\int\int\frac{dB(t)}{dt}\, .\, ds$

$e = \oint E\, dl = \iint \nabla \times E\, ds$ by Stoke’s theorem

$\therefore \iint \nabla \times E\, ds = -\int\int\frac{dB(t)}{dt}\, .\, ds$

$\therefore \nabla \times E = -\frac{dB}{dt}$ 3

Ampere’s Law

$\oint H\, .\, dl = i$

$i = \iint J\, ds = \oint H\, dl = \iint(\nabla \times H).ds$

$\therefore \nabla \times H = J$

By analogy with equ. 3

$\nabla \times H = \frac{\partial D}{\partial t} = J_d$ Displacement current introduced by Maxwell

∇x H = J + J_d = J + $\frac{\partial D}{\partial t}$ 4

The Wave Equation

∇x E = $-\frac{\partial B}{\partial t}$ = -μ $\frac{\partial H}{\partial t}$

Faraday's Law

∇ x ∇ x E = ∇(∇ · E) - ∇^2 E = ∇ x (-μ $\frac{\partial H}{\partial t}$) = -$\mu\frac{\partial}{\partial t}$($\nabla$ x H)

Assume empty space ρ_v = 0 and J = 0

i.e. ∇ · E = 0

$\therefore$ ∇^2 E = $\mu\frac{\partial}{\partial t}$($\nabla$ x H)

= -$\mu\frac{\partial}{\partial t}$(J + $\frac{\partial D}{\partial t}$)

= -$\mu\frac{\partial}{\partial t}$($\frac{\partial D}{\partial t}$)

= -$\mu\epsilon\frac{\partial}{\partial t}$($\frac{\partial E}{\partial t}$)

∇^2 E = $\mu\epsilon\frac{\partial^2 E}{\partial t^2}$

Assume a wave polarised in x direction travelling in z direction with velocity c.

i.e. E_x = f(z − ct) E_y = 0 E_z = 0

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2}$$

$\frac{\partial E_x}{\partial z}$ = f^1(z - ct) $\frac{\partial^2 E_x}{\partial z^2}$ = f^2(z - ct)

$\frac{\partial E_x}{\partial t}$ = -c f^1(z - ct) $\frac{\partial^2 E_x}{\partial t^2}$ = c^2 f^2(z - ct)

f^2(z - ct) = $\mu\epsilon c^2 f^2$(z - ct)

$$c = \frac{1}{\sqrt{\mu\epsilon}}$$